Int. J. Solids Structures, 1969, Vol. 5, pp. 51 to 64. Pergamon Press. Printed in Great Britain

COMBINED DISTRIBUTED LOADS ON RIGID–PLASTIC CIRCULAR PLATES WITH LARGE DEFLECTIONS

NORMAN JONES*

Division of Engineering, Brown University, Providence, Rhode Island

Abstract—A theoretical investigation into the influence of geometry changes on the behavior of a simply supported circular rigid, perfectly plastic plate subjected to two independent distributed pressures is presented herein. The results indicate, as might be anticipated, that when finite deflections are considered such plates could support loads greater than the corresponding collapse pressures obtained recently by Flügge and Gerdeen [4] and Hodge and Sun [6]. The general procedure reported could be used to study the reserve strength of circular plates having different boundary conditions and other kinds of external loads, and could be developed further in order to examine the influence of geometry changes on other rigid–plastic structures subjected to time-dependent or time-independent loads.

NOTATION

H	plate thickness
M_0	$\sigma_0 H^2/4$
M_r, M_θ	radial and circumferential bending moments per unit length
N_0	$\sigma_0 H$
N_r, N_θ	radial and circumferential membrane forces per unit length
Q	transverse shear force per unit length of plate
R	outside radius of plate
W_0	deflection at center of circular plate
a	outer radius of surface area of plate over which pressure p acts as shown in Fig. 1
b, c, d	radii of various zones within the plate which are described by different parts of the yield surface
k	uniform distributed pressure per unit area of undeformed plate
m_r, m_θ	dimensionless bending moments M_r/M_0, M_θ/M_0
n_r, n_θ	dimensionless membrane forces N_r/N_0, N_θ/N_0
p	uniform distributed pressure per unit area of undeformed plate indicated in Fig. 1
$\bar{p}$	p/p_0
p_0	$6M_0/R^2$
q	uniform distributed pressure per unit area of undeformed plate indicated in Fig. 1
$\bar{q}$	q/p_0
r	radial coordinate of plate
s	$-k\cos\phi$
t	$-k\sin\phi$
u	displacement in direction r of undeformed plate
w	transverse deflection perpendicular to undeformed plate
β	b/R
δ	c/R
ε	a/R
$\varepsilon_r, \varepsilon_\theta$	radial and circumferential strains
η	ρ/R
θ	circumferential coordinate lying in plate
κ_r, κ_θ	radial and circumferential curvatures
ρ	radius of central zone in circular plate as defined in section 3
σ_0	yield stress in simple tension
ϕ	slope of the mid-plane of a plate measured in a plane which passes through $r = 0$ and is perpendicular to the plate surface
Δ	W_0/H

* Now at: School of Engineering, Massachusetts Institute of Technology, Boston, Massachusetts.

$(\dot{\ })$ $\qquad \dfrac{\partial}{\partial t}(\ )$

$(\ )'$ $\qquad \dfrac{\partial}{\partial r}(\ )$

1. INTRODUCTION

Drucker and Hopkins [1] studied the collapse of ideally-plastic circular plates subjected to a combination of concentrated and distributed loads. This analysis, which was an extension of Hopkins and Prager's [2] earlier work, was formulated using equilibrium equations derived for the undeformed configuration of the plate, and therefore neglected the influence of membrane forces which arise during deformation. Onat and Haythornthwaite [3] measured the load carrying capacity of initially flat circular mild steel plates and observed that the bending only solution of Hopkins and Prager [2] underestimated considerably the load which could be supported if deflections of the order of the plate thickness or larger were permitted. In order to explain this strengthening effect, Onat and Haythornthwaite incorporated both bending moments and membrane forces in an analysis based on the upper bound theorem of plasticity.

Recently, Flügge and Gerdeen [4] and Hodge and Sun [5] have studied the original problem posed by Drucker and Hopkins [1] and determined the collapse loads for a circular plate which was subjected to a combination of uniformly distributed loads p and q. Clearly there are an infinite number of combinations of p and q which will cause collapse of the plate, and these can be shown to form a convex interaction curve in the p–q plane. Combinations of p and q which lie inside this curve are safe, while those lying on the curve itself would indicate incipient collapse. Values of p and q are not permitted to fall outside the interaction curve for a rigid, perfectly plastic material. It may be shown that there are seven different collapse mechanisms for the plate in the entire p–q plane. Flügge and Gerdeen [4] obtain some of these mechanisms by allowing portions of the plate to remain rigid for which the corresponding generalized stresses therefore lie inside the yield surface, while Hodge and Sun employ a mode vector as described in Ref. [6].

Now, if geometry changes of the plate during deformation are considered, then it is clear that the plate behavior can be described by families of interaction curves in the p–q plane, each family of which is related to a distinct spatial character of the loads, while each curve within a family corresponds to a specific value of the deflection of the plate measured at a convenient location. At a corner of the interaction curve, however, there is the possibility that the deflection of the plate may not be unique, as observed by Flügge and Gerdeen [4] for the case when finite deformations are suppressed.

It is the object of this article to extend the recent work of Flügge and Gerdeen [4] and Hodge and Sun [5, 6] on the behavior of a rigid perfectly plastic circular plate, loaded as indicated in Fig. 1, in order to study situations when finite deflections are permitted.

2. BASIC RELATIONS

The equations of equilibrium in the tangential and transverse directions of the deformed element shown in Fig. 2 may be written in the form [7, 8]

$$rn_r' + n_r - n_\theta = -rkw'/N_0 \tag{1}$$

and

$$rm_r'' + 2m_r' - m_\theta' - 4n_\theta w'/H = rk/M_0. \tag{2}$$

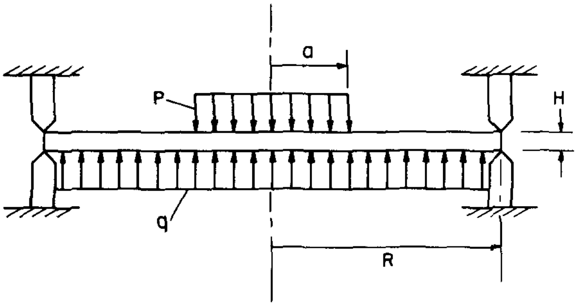

FIG. 1. Simply supported circular plate loaded with two independent pressures p and q.

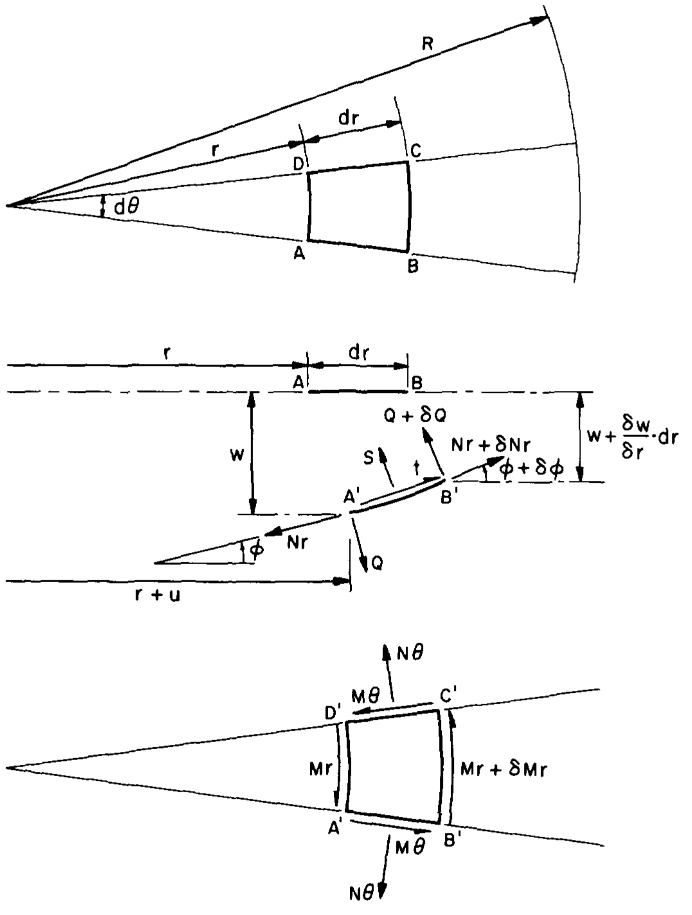

FIG. 2. Element of a circular plate.

If the radial displacement $u = 0$, then the appropriate strain and curvature rates are

$$\dot{\varepsilon}_r = w'\dot{w}' \quad \text{and} \quad \dot{\varepsilon}_\theta = 0 \tag{3}$$

and

$$\dot{\kappa}_r = \dot{w}'' \quad \text{and} \quad \dot{\kappa}_\theta = \dot{w}'/r. \tag{4}$$

In order to achieve an analytical solution to the problem posed, the piecewise linear yield surface proposed by Hodge [9] for a rigid, perfectly plastic material will be used in the following work.

3. SINGLE DISTRIBUTED PRESSURE $p \geq 0$ FOR $0 \leq r \leq a$ AND $q = 0$

In this section let us consider in some detail the particular case of a rigid, perfectly plastic circular plate simply supported around its outer edge and loaded with a pressure p distributed uniformly within a central circular zone of radius a. The bending only solution for this problem may be obtained in a straightforward manner by allowing a plastic hinge to form at the center of the plate. It appears reasonable for finite deflections, therefore, to attempt a solution which allows such a plastic hinge to grow into a central circular zone, the radius ρ of which would increase with increase in deflection and load. Thus, the plate can be analyzed conveniently in three separate regions: $0 \leq r \leq \rho$, $\rho \leq r \leq a$ and $a \leq r \leq R$, viz.

3.1 $0 \leq r \leq \rho$

Now, since the membrane forces and bending moments are axisymmetrically distributed and $p > 0$, then

$$m_r = m_\theta = -1 \quad \text{and} \quad n_r = n_\theta = 1 \tag{5}$$

at the plate center.

If one considers the entire region $0 \leq r \leq \rho$ in this state, then it is necessary that

$$w' = -\frac{rp}{N_0}$$

in order for the equilibrium equation (2) to be satisfied. Thus,

$$w = W_0 - \frac{pr^2}{2N_0}, \tag{6}$$

where W_0 is the maximum deflection at $r = 0$.

This solution can be shown to satisfy the other equilibrium equation (1) provided terms of order ε_r are neglected when compared with unity, an assumption which is consistent with those implicit in the derivation of equations (1) and (2).

If all time rates of quantities are replaced by their derivatives with respect to W_0, then using equations (3) and (4) and noting that $\mathrm{d}p/\mathrm{d}W_0 \geq 0$, it may be shown that $\dot{\varepsilon}_r \geq 0$, $\dot{\varepsilon}_\theta = 0$, $\dot{\kappa}_r \leq 0$ and $\dot{\kappa}_\theta \leq 0$ which are consistent with the normality requirements associated with the selected part of the yield surface [9].

3.2 *Displacement profile*

If a linear displacement profile is selected for the region $\rho \leq r \leq R$, then

$$w = \left(W_0 - \frac{p\rho^2}{2N_0}\right)\left(\frac{R-r}{R-\rho}\right) \tag{7}$$

since the deflection given by equation (7) must match that predicted by equation (6) at $r = \rho$, and $w = 0$ at $r = R$.

Furthermore, if w' is made continuous between the two regions at $r = \rho$, then it can be shown that

$$W_0 = \frac{p\rho R}{N_0} - \frac{p\rho^2}{2N_0},$$

which gives

$$w = \frac{p\rho}{N_0}(R-r) \quad \text{for } \rho \leq r \leq R \tag{8}$$

and

$$\Delta = \tfrac{3}{4}\bar{p}\eta(2-\eta), \tag{9}$$

where

$$\Delta = W_0/H$$

$$\bar{p} = p/p_0$$

$$p_0 = 6M_0/R^2$$

and

$$\eta = \rho/R.$$

3.3 $\rho \leq r \leq a$

If the part of the yield surface [9] described by the equations

$$-1 \leq m_r \leq 0, \qquad m_\theta = -1, \qquad n_r = n_\theta = 1 \tag{10}$$

is selected for this region, then equations (2) and (8) give

$$\frac{\partial}{\partial r}(r^2 m_r') = \frac{pr^2}{M_0} - \frac{p\rho r}{M_0}, \tag{11}$$

the solution of which is

$$m_r = \frac{p\rho^2}{2M_0} - \frac{p\rho r}{2M_0} - \frac{p\rho^3}{6M_0 r} + \frac{pr^2}{6M_0} - 1, \tag{12}$$

where the constants of integration have been determined from the conditions $m_r = -1$ and $m_r' = 0$ at $r = \rho$, which are demanded by the equilibrium requirements across the circular boundary between the two zones.

3.4 $a \leq r \leq R$

If the yield condition (10) and the displacement profile (8) are substituted into equation (2), then it may be shown that

$$\frac{\partial}{\partial r}(r^2 m_r') = -\frac{p\rho r}{M_0},$$

from which

$$m_r = -\frac{p\rho r}{2M_0} - \frac{p\rho^3}{6M_0 r} - \frac{pa^3}{3M_0 r} + \frac{pa^2}{2M_0} + \frac{p\rho^2}{2M_0} - 1, \tag{13}$$

where the constants of integration have been evaluated from the requirements that m_r and m_r' are continuous across the circular boundary between the two regions at $r = a$.

Finally, the requirement that $m_r = 0$ at $r = R$ gives

$$\bar{p}\{\varepsilon^2(3-2\varepsilon) - \eta(3-3\eta+\eta^2)\} - 1 = 0 \tag{14}$$

where $\varepsilon = a/R$.

Equation (14) reduces to

$$\eta = 1-(1/\bar{p})^{\frac{1}{3}} \tag{15}$$

when $\varepsilon = 1$, and

$$\bar{p} = \frac{1}{\varepsilon^2(3-2\varepsilon)}, \quad \text{when } \eta = \Delta = 0, \tag{16}$$

the value corresponding to the appropriate bending only solution of Refs. [4] and [5].

It may be shown that $m_r' \geq 0$ at $r = a$ provided

$$\eta^3 + 2\varepsilon^3 - 3\eta\varepsilon^2 \geq 0$$

which entails no restriction since $0 \leq \eta \leq \varepsilon \leq 1$, while the more limiting expression

$$\eta^3 + 2\varepsilon^3 - 3\eta \geq 0 \tag{17}$$

must be satisfied in order to ensure $m_r' \geq 0$ at $r = R$.

The solution to the posed problem given by equations (9) and (14) subject to the restrictions (17) satisfies the equilibrium equations, boundary conditions, continuity requirements and lies everywhere on the yield surface with associated strain and curvature vectors consistent with normality. Thus the solution is exact for the yield surface selected [9] within the framework of rigid, perfectly plastic theory which has been developed for structures not undergoing geometry changes during deformation.

The results predicted by equations (9) and (15) are plotted in Fig. 3(a) and compared with equation (5) of Ref. [3].

When equation (17) is not satisfied, then it is necessary to consider an additional outer zone in order to ensure that $m_r' \geq 0$ at $r = R$. Such an analysis is a straightforward extension of the work in this section, but it is not presented here because the more general case of two combined pressures p and q will be considered next.

4. COMBINED DISTRIBUTED PRESSURES p AND q; $p \geq 0$ AND $q \geq 0$

In this section we will consider the influence of geometry changes on the behavior of a rigid, perfectly plastic circular plate simply supported around its outer edge and sub-

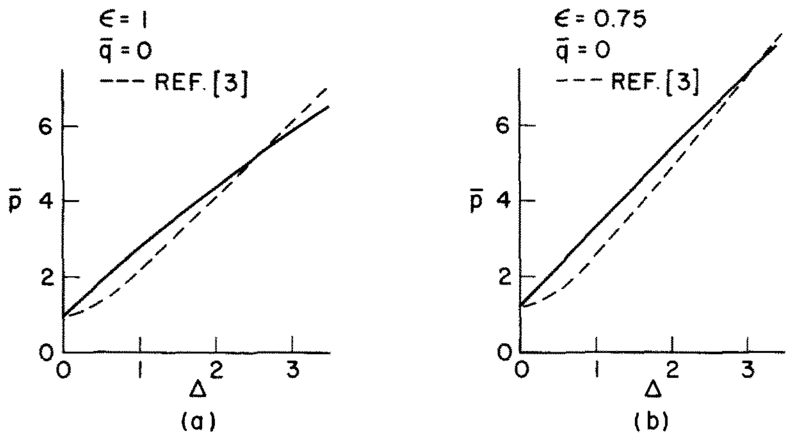

FIG. 3. Comparison of results when $q = 0$ with those of Ref. [3].

jected to the combination of uniformly distributed pressures shown in Fig. 1. It was indicated in section 3 that an additional outer zone must be considered when one is interested in studying deflections, the magnitude of which would cause the inequality (17) to be violated. In fact it is necessary to consider seven different cases or modes of deformation in order to complete that portion of the convex interaction curve which lies in the positive quadrant of the p–q plane. The basic ideas which have already been developed in section 3 can be extended easily to the loading situation under review, viz.

4.1 *Case* 1 : $p > q$

When the magnitude of q is small, one might expect this analysis to be somewhat similar to that considered already in section 3. Indeed, if the same portion of the yield surface is used, it can be shown that

$$\Delta = \tfrac{3}{4}(\bar{p}-\bar{q})\eta(2-\eta) \tag{18}$$

and

$$(\bar{p}-\bar{q})(-3\eta+3\eta^2-\eta^3)+\bar{p}\varepsilon^2(3-2\varepsilon)-\bar{q}-1 = 0, \tag{19}$$

where

$$\bar{q} = q/p_0,$$

provided

$$\eta^3+2\varepsilon^3-3\eta\varepsilon^2 \geq 0 \tag{20}$$

and

$$(\bar{p}-\bar{q})\eta(\eta^2-3)+2\bar{p}\varepsilon^3-2\bar{q} \geq 0. \tag{21}$$

Equations (18) and (19) reduce to equations (9) and (14) when $\bar{q} = 0$.

4.2 *Case* 2 : $p > q$

Case 1 no longer can be used for parameters which do not satisfy the inequality (21) since the yield condition will be violated at the outer edge of the plate. Thus an additional outer zone must be considered, and for the purpose of this analysis the plate may be divided conveniently into the three zones,

$$0 \leq r \leq \rho \quad \text{in which } m_r = m_\theta = -1, \qquad n_r = n_\theta = 1,$$

$$\rho \leq r \leq b \quad \text{in which } -1 \leq m_r \leq 0, \qquad m_\theta = -1, \qquad n_r = n_\theta = 1,$$

and

$$b \leq r \leq R \quad \text{throughout which } m_r - m_\theta = 1, \qquad n_r = n_\theta = 1,$$

where it is assumed that $0 \leq \rho \leq a \leq b \leq R$.

If a procedure somewhat similar to that developed previously is followed, and a logarithmic displacement profile is used in the outer region $b \leq r \leq R$, then it may be shown that

$$\Delta = \tfrac{3}{4}(\bar{p}-\bar{q})\eta\{2\beta(1-\log\beta)-\eta\} \tag{22}$$

$$(\bar{p}-\bar{q})\eta\log\beta\left(\frac{\eta^2}{\beta}-3\beta+3\beta\log\beta\right)+\tfrac{3}{2}\bar{q}(1-\beta^2+\tfrac{2}{3}\beta^2\log\beta)+\frac{2\bar{p}\varepsilon^3}{\beta}\log\beta = 0 \tag{23}$$

and

$$(\bar{p}-\bar{q})\eta\left(3\eta-\frac{\eta^2}{\beta}-3\beta\right)-q\beta^2+\bar{p}\varepsilon^2\left(3-\frac{2\varepsilon}{\beta}\right)-1 = 0 \tag{24}$$

where

$$\beta = b/R,$$

provided

$$\eta^3+2\varepsilon^3-3\eta\varepsilon^2 \geq 0, \tag{25}$$

$$(\bar{p}-\bar{q})\eta(\eta^2-3\beta^2)+2\bar{p}\varepsilon^3-2\bar{q}\beta^3 \geq 0 \tag{26}$$

and

$$(\bar{p}-\bar{q})\eta(3\beta^2-\eta^2-6\beta^2\log\beta)+\bar{q}\beta(3-\beta^2)-2\bar{p}\varepsilon^3 \geq 0. \tag{27}$$

Equations (22)–(27) reduce to equations (18)–(21) when $\beta = 1$ for which equation (21) is an equality.

4.3 *Case* 3: $p > q$

It may be shown that an extension of Case 2 to accommodate situations when $0 \leq \rho \leq b \leq a \leq R$ gives

$$\Delta = \tfrac{3}{4}(\bar{p}-\bar{q})\eta\{2\beta(1-\log\beta)-\eta\}, \tag{28}$$

$$(\bar{p}-\bar{q})\left\{\beta^2(\log\beta-\tfrac{3}{2})+3\eta\beta\log\beta(1-\log\beta)-\frac{\eta^3}{\beta}\log\beta\right\}+\tfrac{3}{2}\bar{p}\varepsilon^2(1-2\log\varepsilon)-\tfrac{3}{2}\bar{q} = 0 \tag{29}$$

and

$$(\bar{p}-\bar{q})\left(3\eta^2-3\eta\beta-\frac{\eta^3}{\beta}+\beta^2\right)-1 = 0, \tag{30}$$

provided

$$\eta^3+2\beta^3-3\eta\beta^2 \geq 0, \tag{31}$$

and

$$(\bar{p}-\bar{q})(\beta^3+3\eta\beta^2-\eta^3-6\eta\beta^2\log\beta)+3\beta(\bar{q}-\bar{p}\varepsilon^2) \geq 0. \tag{32}$$

4.4 *Case* 4: $p > q$

The interaction curve for positive deflections and $p > q$ may be extended further by allowing an outer annular portion of the plate to remain rigid. Thus, employing the piecewise linear yield surface [9], the plate can be divided into the three zones,

$$0 \leq r \leq \rho \quad \text{throughout which } m_r = m_\theta = -1, \qquad n_r = n_\theta = 1,$$

$$\rho \leq r \leq b \quad \text{described by } -1 \leq m_r \leq 0, \qquad m_\theta = -1, \qquad n_r = n_\theta = 1,$$

and

$$b \leq r \leq c \quad \text{in which } m_r - m_\theta = 1 \quad \text{and} \quad n_r = n_\theta = 1.$$

The circular boundary of radius c is at the point $m_r = 1$, $m_\theta = 0$ and $n_r = n_\theta = 1$ of the yield surface [9] and the portion $c \leq r \leq R$ of the plate remains rigid and is, therefore, described by a combination of moments and forces which are allowed to lie within the yield surface. Clearly the deflection of the plate in the region $c \leq r \leq R$ must be zero, but the moments and forces cannot have unique values if the plate is made from a rigid, perfectly plastic material. However, one possible relationship between the forces and moments is

$$0 \leq m_r \leq 1, \qquad m_\theta = 0, \qquad n_r = n_\theta = 1$$

in the zone $c \leq r \leq d$, and

$$m_r - m_\theta = m_1, \qquad n_r = n_\theta = 1 \quad \text{and} \quad 0 \leq m_1 \leq 1$$

throughout the region $d \leq r \leq R$.

Thus, solving the equilibrium equations and satisfying the boundary conditions and continuity requirements when using the aforementioned generalized stress profiles gives

$$\Delta = \tfrac{3}{4}(\bar{p}-\bar{q})\eta(2\beta\log\delta/\beta+2\beta-\eta), \tag{33}$$

$$(\bar{p}-\bar{q})\left(-3\eta\beta+3\eta^2-\frac{\eta^3}{\beta}+\beta^2\right)-1 = 0 \tag{34}$$

$$(\bar{p}-\bar{q})\left(3\varepsilon^2-\beta^2-3\eta\beta+\frac{\eta^3}{\beta}-6\eta\beta\log\delta/\beta\right)-3\bar{q}(\delta^2-\varepsilon^2) = 0, \tag{35}$$

and

$$\begin{aligned}(\bar{p}-\bar{q})\left\{3\eta\beta((\log\delta/\varepsilon)^2-(\log\varepsilon/\beta)^2)+\log\varepsilon/\beta\left(-3\eta\beta-\beta^2+\frac{\eta^3}{\beta}\right)-\tfrac{3}{2}\beta^2\right\}\\ -\tfrac{3}{2}\bar{q}\delta^2(1-2\log\delta/\varepsilon)+\tfrac{3}{2}\bar{p}\varepsilon^2-1 = 0\end{aligned} \tag{36}$$

where

$$\delta = c/R,$$

provided

$$\eta^3+2\beta^3-3\eta\beta^2 \geq 0 \tag{37}$$

$$\eta^3+3\varepsilon^2\beta-\beta^3-3\eta\beta^2(1+2\log\varepsilon/\beta) \geq 0. \tag{38}$$

The suggested profile in the outer rigid zone gives

$$m_r = 1-\frac{\bar{q}r^2}{R^2}+3\bar{q}\delta^2-\frac{2\bar{q}\delta^3R}{r} \quad \text{for } c \leq r \leq d \tag{39}$$

and

$$m_r = -\frac{3}{2}\frac{\bar{q}r^2}{R^2}+\tfrac{3}{2}\bar{q}+(3\bar{p}\varepsilon^2-m_1)\log(r/R) \quad \text{for } d \leq r \leq R. \tag{40}$$

Continuity of m_r at $r = d$ gives one equation between the unknowns d and m_1. Thus the solution as in Ref. [4] is arbitrary to this degree.

4.5 *Case* 5: $q > p$

One might expect that this case could be analyzed in a manner somewhat similar to Case 1 with $\varepsilon = 1$ when p is small. In fact using the following portions of the yield surface [9]

$$m_r = m_\theta = n_r = n_\theta = 1 \quad \text{for } 0 \leq r \leq \rho,$$

and

$$0 \leq m_r \leq 1, \qquad m_\theta = 1, \qquad n_r = n_\theta = 1 \quad \text{for } \rho \leq r \leq R,$$

one can show that

$$\Delta = -\tfrac{3}{4}(\bar{q}-\bar{p})\eta(2-\eta) \tag{41}$$

and

$$(\bar{q}-\bar{p})\eta(3-3\eta+\eta^2)+\bar{p}\varepsilon^2(3-2\varepsilon)-\bar{q}+1 = 0 \tag{42}$$

provided

$$\eta^3+2\varepsilon^3-3\eta\varepsilon^2 \geq 0, \tag{43}$$

and

$$(\bar{q}-\bar{p})\eta(\eta^2-3)+2\bar{q}-2\bar{p}\varepsilon^3 \geq 0. \tag{44}$$

$|\Delta|$ now represents the magnitude of the vertically upwards deflection at the center of the plate.

4.6 *Case* 6: $p > q$

Even though $p > q$ we will now examine the situation when the applied loads cause the plate to deflect upwards. Thus using the portion of the yield surface [9] represented by the equations

$$m_r = 1, \qquad 0 \leq m_\theta \leq 1, \qquad n_r = n_\theta = 1 \quad \text{for } 0 \leq r \leq b$$

and

$$0 \leq m_r \leq 1, \qquad m_\theta = 1, \qquad n_r = n_\theta = 1 \quad \text{for } b \leq r \leq R,$$

it may be shown that

$$\bar{q}\beta^2 = \bar{p}\varepsilon^2 \tag{45}$$

and

$$-2\Delta(1-\beta)-\bar{q}(1-3\beta^2+2\beta^3)+1 = 0, \tag{46}$$

provided

$$\bar{p}-\bar{q} \leq 1/3\varepsilon^2 \tag{47}$$

and

$$-\Delta \leq \bar{q}(1-\beta^3)/(1+\beta) \tag{48}$$

In common with the work in refs. [4, 5], it is necessary, in order to obtain a solution for this case, to permit a discontinuity in dw/dr across the circular boundary between the two regions at $r = b$.* However, if ε is given and one assigns particular values to $\bar{p}$ and $\bar{q}$, then β can be calculated from (45) and the corresponding value of Δ obtained from (46).

4.7 *Case* 7

If the inequality (47) is not satisfied, then it is necessary to allow an inner zone $0 \leq r \leq \rho$ of the plate to remain rigid while for

$$\rho \leq r \leq b, \qquad m_r = 1, \qquad m_1 \leq m_\theta \leq 1, \qquad n_r = n_\theta = 1,$$

and

$$0 \leq m_r \leq 1, \qquad m_\theta = 1, \qquad n_r = n_\theta = 1 \quad \text{for } b \leq r \leq R.$$

One admissible generalized stress profile in the rigid portion is

$$m_\theta = -m_2, \qquad -m_2 \leq m_r \leq m_3, \qquad 0 \leq m_2 \leq 1, \qquad 0 \leq m_3 \leq 1$$
$$n_r = n_\theta = 1, \quad \text{for } 0 \leq r \leq a,$$

and

$$m_r - m_\theta = 1 - m_1, \qquad n_r = n_\theta = 1 \quad \text{for } a \leq r \leq \rho,$$

which gives, finally, equations (45), (46) and (48) with

$$\beta^2 - \eta^2 \leq 1/3\bar{q} \tag{49}$$

in order to maintain $m_1 \leq 1$.

The equations for the stress profile in the rigid region $0 \leq r \leq \rho$ of the plate are the same as those used by Flügge and Gerdeen [4] for the collapse mode III. These may be shown to give finally one equation relating the parameters η and m_2 for continuity of m_r at $r = a$. Thus as indicated in reference [4] and earlier for Case 4 the solution is not unique in the rigid zone.

5. DISCUSSION

The various equations presented in sections 3 and 4 reduce when $\Delta = 0$ to the corresponding bending only solution of Flügge and Gerdeen [4], while the results for

* It might be noted here that dw/dr should be continuous across a travelling hinge when finite deflections are permitted.

Cases 1, 2, 3, 5 and 6 with $\Delta = 0$ are identical to those in reference [5]. In Cases 4 and 7 outlined in sections 4.4 and 4.7 portions of the plate were allowed to remain rigid in a manner similar to that developed by Flügge and Gerdeen [4] while as mentioned previously, Hodge and Sun [6] employed a mode vector approach in order to complete the interaction curve.

The predictions of the theoretical analysis outlined here are plotted in Figs. 3(a) and 3(b) for the particular cases in which $\varepsilon = 1$ and $\varepsilon = 0{\cdot}75$ when $\bar{q} = 0$ and compared with the approximate upper bound solution of Onat and Haythornthwaite [3]. The latter [3] disregarded any radial strain which might arise during deformation and allowed the plate to slide over the supports.

It is clear from Fig. 4, which is plotted using the predictions of section 4, that the interaction curve expands with increase in the allowable center deflection (Δ) of the plate.

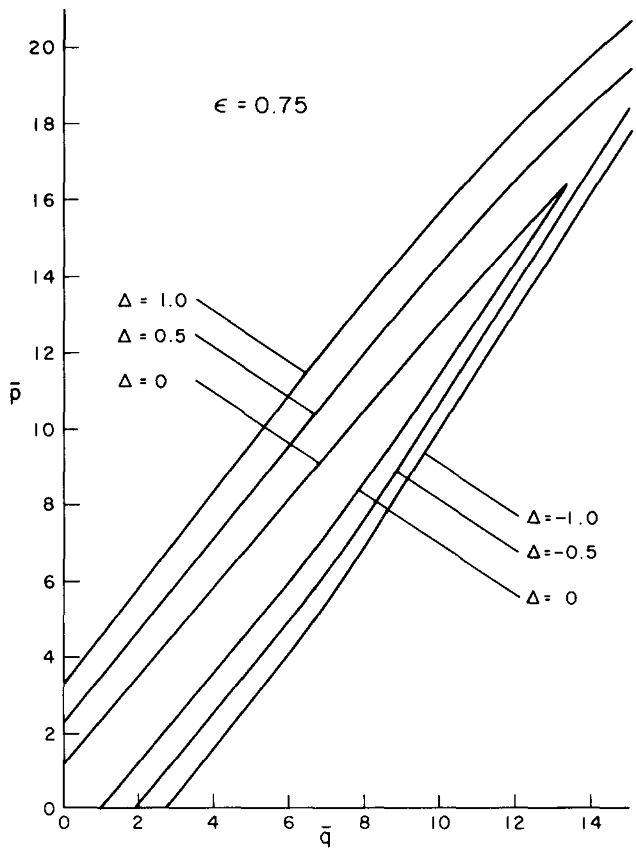

FIG. 4. The influence of finite-deflections on the interaction curve for a simply supported rigid–plastic circular plate subjected to two independent pressures p and q.

This phenomenon, as discussed previously in section 1, is due to the action of membrane forces which are introduced when changes in geometry occur. The plate remains rigid for combinations of the loads $\bar{p}$ and $\bar{q}$ which lie within the convex interaction curve $\Delta = 0$, while combinations lying on the curve $\Delta = 0$ indicate incipient collapse according to the bending only theory [4, 5]. It is relevant to remark here that the interaction curves for $\Delta = \Delta_1$, where $\Delta_1 > 0$ imply, of course, that upon removal of the loads which produced a deflection $\Delta = \Delta_1$, the plate remains permanently deformed with a maximum deflection of the magnitude Δ_1. Upon reloading along the previous proportional loading path, however, the plate remains rigid until subjected to the same combination of loads which produced originally the deflection Δ_1. It is important to emphasize here that the curves $\Delta > 0$ should be thought of as initial proportional loading curves only because reloading introduces complications for any other than the simple case just discussed.

Now all the various solutions presented here satisfy the equilibrium equations, boundary conditions, continuity requirements and have generalized stresses which lie everywhere on or within the yield surface, with associated strain-rate and curvature-rate vectors consistent with the requirements of normality. Thus the solutions are exact for the yield surface selected [9] within the framework of the rigid, perfectly plastic theory which has been developed for structures not undergoing geometry changes during deformation. However, the same comments noted in the conclusion of Onat and Haythornthwaite [3] concerning the fact that the assumed shape of the deflected plate may not coincide with the most favorable profile developed by an actual plate apply equally here. Furthermore, in order to make the problem tractable, the piecewise linear four dimensional yield surface of Hodge [9] was used rather than the exact one of Onat and Prager [10]. This approximate yield surface circumscribes the exact one, while another 0·618 times as large would inscribe it. It is assumed that the true behavior lies somewhere between the solutions obtained from these two yield surfaces. Intuitively, one might expect this to be reasonable, but there is no guarantee that the displacement at a point on a structure does lie between the results obtained from these two yield surfaces because a linearization of the yield surface invariably restricts response due to the normality requirements. Naturally, these latter comments apply equally well to all analyses using piecewise linear approximations of the exact yield surface.

It was assumed when deriving the sections of the interaction curve described by Cases 4 and 7 that certain portions of the plate remained rigid and were therefore described by combinations of generalized stresses which lay within the yield surface. Clearly, when a plate is made from a rigid, perfectly plastic material, the generalized stresses which lie within the rigid zones do not have unique values. It may be necessary for certain parameters to use different generalized stress profiles within the rigid zones to those suggested in sections 4.4 and 4.7. Nevertheless the equations for the interaction curves would remain unchanged.

The importance of material strain-hardening, for the cases in which no hinges form, could be examined in a manner similar to that developed in ref. [11]. However, the influence of this additional parameter might be expected to be small because of the small strain assumptions incorporated in the present theory.

6. CONCLUSIONS

A theoretical study of the behavior of a simply supported rigid, perfectly plastic circular plate subjected to two independent distributed pressures is presented here for the case when geometry changes are permitted. The results indicate clearly that the plate could support pressures greater than those obtained recently by Flügge and Gerdeen [4] and Hodge and Sun [5, 6] when the influence of geometry changes is disregarded.

It should prove straightforward to extend the work reported here in order to examine the reserve strength of plates with different boundary conditions and subjected to other kinds of external loads. In fact this general procedure has been used already to study the dynamic behavior of rigid–plastic circular plates [8] and could be developed further in order to examine the influence of finite-deflections on the behavior of other rigid–plastic structures subjected to time-dependent or time-independent loads.

Acknowledgements—The work reported herein was supported by the National Science Foundation under Grant Number GK-1013 awarded to Brown University. The author wishes to take this opportunity to express his appreciation to Miss E. Cerutti for computing the final results and to Miss M. Wheat for typing the manuscript.

REFERENCES

[1] D. C. DRUCKER and H. G. HOPKINS, Combined concentrated and distributed load on ideally-plastic circular plates. *Proc. 2nd U.S. natn. Congr. appl. Mech.*, 517–520 (1954).
[2] H. G. HOPKINS and W. PRAGER, The load carrying capacities of circular plates. *J. Mech. Phys. Solids* **2**, 1–13 (1953).
[3] E. T. ONAT and R. M. HAYTHORNTHWAITE, The load carrying capacity of circular plates at large deflection. *J. appl. Mech.* **23**, 49–55 (1956).
[4] W. FLÜGGE and J. C. GERDEEN, Collapse of a simply supported circular plate under two uniform loads. *Int. J. Solids Struct.* **3**, 677–689 (1967).
[5] P. G. HODGE and C. K. SUN, Yield-point load of a circular plate sealing an incompressible fluid. *Int. J. mech. Sci.* **9**, 405–414 (1967).
[6] P. G. HODGE and C. K. SUN, General properties of yield-point load surfaces. *J. appl. Mech.* **35**, 107–110 (1968).
[7] E. REISSNER, *Proc. Symp. appl. Math.* **1**, 213–219 (1949).
[8] N. JONES, Finite deflections of a simply supported rigid–plastic annular plate loaded dynamically. *Int. J. Solids Struct.* **4**, 593–603 (1968).
[9] P. G. HODGE, Yield conditions for rotationally symmetric shells under axisymmetric loading, *J. appl. Mech.* **27**, 323–331 (1960).
[10] E. T. ONAT and W. PRAGER, Limit analysis of shells of revolution, Parts I and II, *Proc. R. Ned. Acad. Sci.* **B57**, 534–541 and 542–548 (1954).
[11] N. JONES, Finite deflections of a rigid viscoplastic strain-hardening annular plate loaded impulsively. *J. appl. Mech.* **35**, 349–356 (1968).

(*Received* 4 *March* 1968)

Абстракт—Приводится теоретическое исследование влияния изменений геометрии на поведение свободно опертых, круглых, жестко идеально пластических пластинок, подверженных двум независимо распределенным давлениям. Результаты указывают, как можно было ожидать, что при учете конечных прогибов такие пластинки могут переносить большую нагрузку чем соответствующие давления при разрушении, полеченные недавно Флюггем и Джердином [4] и Ходжем и Саном [6]. Указанный общий способ можно использовать для исследования записа прочности круглых пластинок, обладующих разными граничными условиями и другими родами внешных нагрузок. Далее можно этот способ развить для испытания влияния изменений геометрии других жестко-пластиче ских конструкций, подверженных нагрузке, зависящей и независящей от времени.